

Effective field theory and projective construction for Z_k parafermion fractional quantum Hall states

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The projective construction is a powerful approach to deriving the bulk and edge field theories of non-Abelian fractional quantum Hall (FQH) states and yields an understanding of non-Abelian FQH states in terms of the simpler integer quantum Hall states. Here we show how to apply the projective construction to the Z_k parafermion (Laughlin/Moore-Read/Read-Rezayi) FQH states, which occur at filling fraction $\nu=k/(kM+2)$. This allows us to derive the bulk low-energy effective field theory for these topological phases, which is found to be a Chern-Simons theory at level 1 with a $U(M)\times Sp(2k)$ gauge field. This approach also helps us understand the non-Abelian quasiholes in terms of holes of the integer quantum Hall states.

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I. INTRODUCTION

Topological order in the quantum Hall liquids is currently the subject of intense interest because of the possibility of detecting, for the first time, excitations that exhibit non-Abelian statistics.^{1,2} On the theoretical side, a primary issue is how to go beyond some of the known examples of non-Abelian fractional quantum Hall (FQH) states and to construct and understand more general non-Abelian FQH phases.

From the very beginning, two ways to construct and understand non-Abelian FQH states have been developed.^{1,2} One is through the use of ideal-wave functions and ideal Hamiltonians.^{1,3} The physical properties of the constructed FQH states can be deduced using conformal field theory (CFT). The other is the projective construction,^{2,4} which allows us to derive the bulk effective theory and edge effective theory for the constructed FQH states. The physical properties of the FQH states can be derived from those effective theories.

The Z_k parafermion states at filling fraction $\nu=k/(kM+2)$ were first studied using the ideal-wave function/ideal-Hamiltonian approach.^{1,5} What is the bulk effective theory for such Z_k parafermion states? When $M=0$, the edge states of the $\nu=k/2$ Z_k parafermion state are described by the $SU(2)_k$ Kac-Moody (KM) algebra. Using the correspondence between CFT and Chern-Simons (CS) theory,⁶ it was suggested that the bulk effective theory for the $\nu=k/2$ Z_k parafermion state is the $SU(2)_k$ CS theory.^{7,8} The guessed $SU(2)_k$ CS theory correctly reproduces the $(k+1)$ -fold degeneracy for the $\nu=k/2$ Z_k parafermion state on a torus.

However, the $SU(2)_k$ CS theory has a serious flaw. The $SU(2)$ charges in the $SU(2)_k$ KM algebra for the edge states are physical quantum numbers that can be coupled to external probes, while the $SU(2)$ charges in the $SU(2)_k$ CS theory are unphysical and cannot be coupled to external probes without breaking the $SU(2)$ gauge symmetry. This suggests that the $SU(2)$ in the edge $SU(2)_k$ KM algebra is not related to the $SU(2)$ in the bulk $SU(2)_k$ CS theory. This leads us to wonder that the CFT/CS theory correspondence may not be the right way to derive the bulk effective theory for generic non-Abelian states. In fact, when $M\neq 0$, the edge states for

the $\nu=k/(kM+2)$ Z_k parafermion state are described by $U(1)_n\otimes Z_k$ CFT, where the Z_k CFT denotes the Z_k parafermion CFT (Ref. 9) and $n=k(kM+2)/4$.⁸ It is not clear what is the corresponding bulk effective theory. Note that the Z_k parafermion CFT can be obtained from the coset construction of the $SU(2)_k/U(1)$ KM algebra.¹⁰ This suggests that the bulk effective theory may be a $SU(2)_k\otimes U(1)\otimes U(1)$ CS theory.⁸ But a naive treatment of such a CS theory gives rise to $(k+1)\times$ integer number of degenerate ground states on a torus, which does not agree with the ground-state degeneracy for the $\nu=k/(kM+2)$ Z_k parafermion state. We see that the bulk effective theory for a generic parafermion state is still an unresolved issue.

In this paper, we show how the projective construction^{2,4} can be applied to the Z_k parafermion (Read-Rezayi) states. This leads to a simplified understanding of the Z_k parafermion states in terms of the integer quantum Hall (IQH) states and a different way of computing their topological properties. We find the bulk effective theory for the $\nu=k/(kM+2)$ Z_k parafermion state to be the $[U(M)\times Sp(2k)]_1$ CS theory (with a certain choice of electron operators and fermionic cores for certain quasiparticles). Such a CS theory correctly reproduces the ground state degeneracy on a torus.

II. PROJECTIVE CONSTRUCTION

The projective construction was explained in detail in Ref. 4. The idea is to rewrite the electron operator in terms of new fermionic degrees of freedom

$$\Psi_e = \sum_{\{\alpha\}} \psi_{\alpha_1} \cdots \psi_{\alpha_n} C_{\alpha_1 \dots \alpha_n}. \quad (1)$$

There are n flavors of fermion fields, ψ_{α} , for $\alpha=1, \dots, n$, which carry electromagnetic charge q_{α} , respectively, and which are called ‘‘partons.’’ The $C_{\alpha_1 \dots \alpha_n}$ are constant coefficients and the sum of the charges of the partons is equal to the charge of the electron, which we set to 1: $\sum_{\alpha} q_{\alpha}=1$. The electron operator Ψ_e can be viewed as the singlet of a group G , which is the group of transformations on the partons that keeps the electron operator invariant. The theory in terms of electrons can be rewritten in terms of a theory of partons,

provided that we find a way to project the newly enlarged Hilbert space onto the physical Hilbert space, which is generated by electron operators. We can implement this projection at the Lagrangian level by introducing a gauge field, with gauge group G , which couples to the current and density of the partons. We can therefore write the Lagrangian as

$$\mathcal{L} = i\psi^\dagger \partial_0 \psi + \frac{1}{2m} \psi^\dagger (\partial_i - iA_i Q)^2 \psi + \text{Tr}(j^\mu a_\mu) + \dots \quad (2)$$

Here, $\psi^\dagger = (\psi_1^\dagger, \dots, \psi_n^\dagger)$, a is a gauge field in the $n \times n$ matrix representation of the group G . A is the external electromagnetic gauge field and $Q_{ij} = q_i \delta_{ij}$ is an $n \times n$ matrix with the electromagnetic charge of each of the partons along the diagonal. The \dots represent additional interaction terms between the partons and $j_{ab}^\mu = \psi_a^\dagger \partial^\mu \psi_b$. Eq. (2) is simply a convenient rewriting of the theory for the original electron system in terms of a different set of fluctuating fields.

Now we assume that there exists some choice of microscopic interaction parameters for which the interaction between the partons is such that the low-energy fluctuations of the a_μ gauge field are weak after integrating out the partons. This means that the gauge theory that results from integrating out the partons can be treated perturbatively about its free Gaussian fixed point. Since the partons in the absence of the gauge field form a gapped state $|\Phi_{parton}\rangle$ and since we can treat the gauge field perturbatively, the ground state remains to be gapped even after we include the gauge fluctuations. The ground-state wave function is, at least for large separations, $|z_i - z_j| \gg 1$, of the form

$$\Phi(\{z_i\}) = \langle 0 | \prod_{i=1}^N \Psi_e(z_i) | \Phi_{parton} \rangle. \quad (3)$$

If we assume that the i th parton forms a $\nu=1$ integer quantum Hall state, the partons will be gapped and can be integrated out to obtain an effective action solely in terms of the gauge field. The action that we obtain is a CS action with gauge group G , which should be expected given that for a system that breaks parity and time reversal, the CS term is the most relevant term in the Lagrangian at long wavelengths. If we ignore the topological properties of the parton IQH states, then integrating out the partons will yield⁴

$$\mathcal{L} = \frac{1}{4\pi} \text{Tr}(a \partial a) + \frac{1}{2\pi} A \text{Tr}(Q \partial a) + \frac{\text{Tr}(Q^2)}{4\pi} A \partial A + \dots, \quad (4)$$

where $A \partial A = \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$ and the \dots represents higher order terms. Note that the filling fraction of the corresponding FQH state is given by $\nu = \text{Tr}(Q^2) = \sum_a q_a q_a$. Since the partons do not form a trivial gapped state, but rather a topologically nontrivial one, Eq. (4) can only describe ground-state properties of the phase. It can be expected to reproduce the correct result for the ground-state degeneracy on genus g surfaces, for instance, and the correct fusion rules for the non-Abelian excitations, but it cannot be expected to produce all of the correct quantum numbers for the quasiparticle excitations, such as the quasiparticle spin,²¹ unless the partons are treated more carefully. This can be done in two ways. One

way is to not integrate out the partons and to use Eq. (2), taking into account a Chern-Simons term for a_μ that emerges as we renormalize to low energies. As will be discussed in more detail in Sec. V, the quasiparticles will correspond to holes in the parton IQH states which become non-Abelian as a result of the coupling to the non-Abelian Chern-Simons gauge field. The other way is to use the pure gauge theory in Eq. (4) and to put in by hand a fermionic core for quasiparticles that lie in certain ‘‘odd’’ representations of G . Some quasiparticles correspond to an odd number of holes in the parton IQH states and the fermionic character of these odd numbers of holes should be taken into account.

Let us now turn to the edge theory. Before the introduction of the gauge field, the edge theory is the edge theory for n free fermions forming an integer quantum Hall state. If each parton forms a $\nu=1$ IQH state, then the edge theory would be a CFT describing n chiral free fermions, which we will denote as $U(1)^n$. After projection, the edge theory is described by a $U(1)^n/G$ coset theory that we will understand in some more detail when we specialize to the Z_k parafermion states.

To be more precise, the edge theory should be understood in the following way. The electron creation and annihilation operators, Ψ_e and Ψ_e^\dagger , generate an operator algebra that we refer to as the electron operator algebra. Such electron operator algebra can be embedded in the $U(1)^n/G$ coset theory. The topologically distinct quasiparticles are then labeled by different representations of this electron operator algebra. In some cases, the electron operator algebra coincides with some well-known algebra. For the bosonic Z_k parafermion states at $\nu=k/2$, for instance, the electron operator algebra is the same as the $SU(2)_k$ KM algebra, for which the representation theory is well-known.

III. EFFECTIVE THEORY OF PARAFERMION STATES

Now let us apply the projective construction to obtain the Z_k parafermion states. A crucial result for the projective construction is that the $\nu=1$ FQH wave function coincides with the correlation function of free fermions in a 1+1d CFT:

$$\prod_{i<j} (z_i - z_j) = \lim_{z_\infty \rightarrow \infty} z_\infty^{2h_N} \langle e^{-iN\phi(z_\infty)} \psi(z_1) \cdots \psi(z_N) \rangle, \quad (5)$$

where $\psi(z)$ is a free complex chiral fermion, and $\partial\phi = \psi^\dagger \psi$ is the fermion current. The operator product expansions for $\psi(z)$ satisfy $\psi^\dagger(z)\psi(w) \sim \frac{1}{z-w}$ and $\psi(z)\psi(w) \sim (z-w)\psi\partial\psi(w)$. Equation (5) implies that the wave function (3) can also be expressed as a correlation function in a 1+1d CFT (Ref. 4)

$$\Phi(\{z_i\}) = \lim_{z_\infty \rightarrow \infty} z_\infty^{2h_N} \left\langle e^{-iN\phi(z_\infty)} \prod_i \Psi_e(z_i) \right\rangle, \quad (6)$$

where the partons $\psi_i(z)$ are now interpreted as free complex chiral fermions in a 1+1d CFT.

The Z_k parafermion FQH wave functions are constructed as correlation functions of a certain CFT

$$\Phi_{Z_k} = \lim_{z_\infty \rightarrow \infty} z_\infty^{2h_N} \langle e^{-iN\phi(z_\infty)} V_e(z_1) \cdots V_e(z_N) \rangle, \quad (7)$$

where $V_e = \psi_1 e^{i\sqrt{1/\nu}\phi}$. ψ_1 is a simple-current operator in the Z_k parafermion CFT of Zamalodchikov and Fateev⁹ and ϕ is a

free scalar boson. These wave functions exist for $\nu = \frac{k}{kM+2}$; for $M=0$, the electron operator $V_e = \psi_1 e^{i\sqrt{2k}\phi} \equiv J^+$ and $V_e^\dagger = \psi_1^\dagger e^{-i\sqrt{2k}\phi} \equiv J^-$ generate the $SU(2)_k$ KM algebra

$$J^a(z)J^b(0) \sim \frac{k\delta_{ab}}{z^2} + \frac{if_{abc}J^c(0)}{z} + \dots, \quad (8)$$

where $a, b=1, 2, 3$ and $J^\pm = J^1 \pm iJ^2$. This means that any electron operator that satisfies the $SU(2)_k$ current algebra will yield the same wave function. The crucial result for the projective construction approach to the Z_k parafermion states is that if we take the electron operator to be

$$\Psi_{e;k} = \sum_{a=1}^k \psi_{2a-1} \psi_{2a}, \quad (9)$$

then it is easy to verify that $\Psi_{e;k}$ and $\Psi_{e;k}^\dagger$ also satisfy the $SU(2)_k$ current algebra and therefore the wave function (6) is the Z_k parafermion wave function. It follows that the Z_k parafermion states at $\nu = \frac{k}{kM+2}$, for general M , are reproduced in the projective construction for the following choice of electron operator

$$\Psi_e^{(k;M)} = \psi_{2k+1} \cdots \psi_{2k+M} \sum_{a=1}^k \psi_{2a-1} \psi_{2a}, \quad (10)$$

because including the additional operators $\psi_{2k+1}, \dots, \psi_{2k+M}$, each of which is in a $\nu=1$ IQH state, has the effect of multiplying Φ_{Z_k} by the Jastrow factor $\prod_{i<j} (z_i - z_j)^M$.

In the case $M=0$, the electron operator can be written as $\Psi_e^{(k;0)} = \psi^T A \psi$, where $\psi^T = (\psi_1, \dots, \psi_{2k})$ and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.²² I is the $k \times k$ identity matrix. The group of transformations on the partons that leaves the electron operator invariant is simply the group of $2k \times 2k$ matrices that keeps invariant the anti-symmetric matrix A . In this case, this group is the fundamental representation of $Sp(2k)$. Note that $Sp(2) = SU(2)$ and $Sp(4) = SO(5)$. Thus, we expect the edge theory to be $U(1)^{2k}/Sp(2k)_1$, and the bulk CS theory to be $Sp(2k)_1$, as described in the previous section. For general M , the edge theory becomes $U(1)^{2k+M}/[U(M) \times Sp(2k)]_1$ and the bulk effective theory is a $[U(M) \times Sp(2k)]_1$ Chern-Simons theory.

IV. GROUND-STATE DEGENERACY FROM EFFECTIVE CS THEORY

As a first check that this CS theory reproduces the correct topological properties of the Z_k parafermion states, we calculate the ground-state degeneracy on a torus. This can be done explicitly using the methods of Refs. 4 and 11; for $M=0$, the result is $k+1$, which coincides with the torus degeneracy of the $M=0$ Z_k parafermion states. In Appendix A we outline in more detail the calculation in the case $M=1$, for which we find the ground-state degeneracy on a torus to be $(k+1)(k+2)/2$, which also agrees with known results for the Z_k parafermion states.

The case $M=1$ reveals a crucial point. In this case, we have $[U(1) \times Sp(2k)]_1$ CS theory. Naively, we would think that the extra $U(1)_1$ part is trivial and does not contribute to

the ground-state degeneracy or the fusion rules and again we might expect a ground-state degeneracy of $k+1$, but this is incorrect. The reason for this is that usually when we specify the gauge group and the level for CS theory, there is a standard interpretation of what the large gauge transformations are on higher genus surfaces, but this standard prescription may be inapplicable. Instead, the large gauge transformations are specified by the choice of electron operator. In particular, for odd k , the extra factor $(k+2)/2$ is half-integer, which highlights the fact that the $U(1)$ and $Sp(2k)$ parts are married together in a nontrivial way.

In the $M=0$ case, the standard interpretation of the allowed gauge transformations for the $Sp(2k)_1$ CS theory is correct and we can follow the standard prescription for deriving topological properties of CS theories at level k with a simple Lie group G . In these cases, the ground state degeneracy is given by the number of integrable representations of the affine Lie algebra \hat{g}_k , where g is the Lie algebra of G . The quasiparticles are in one-to-one correspondence with the integrable representations of \hat{g}_k , and their fusion rules are identical as well. In the case of the $M=0$ Z_k parafermion states, it is already known that the different quasiparticles correspond to the different integrable representations of the $SU(2)_k$ KM algebra and the fusion rules are the same as the fusion rules of the $SU(2)_k$ representations. In fact, $Sp(2k)_1$ and $SU(2)_k$ have the same number of primary fields and the same fusion rules, and so the $Sp(2k)_1$ CS theory has the same fusion rules as the Z_k parafermion states and the same ground-state degeneracies on high genus Riemann surfaces. The equivalence of the fusion rules for the representations $Sp(2k)_1$ and $SU(2)_k$ current algebra is a special case of a more general ‘‘level-rank’’ duality between $Sp(2k)_n$ and $Sp(2n)_k$,¹² and is also related to the fact that the edge theory for the $M=0$ Z_k parafermion states can be described either by the $U(1)^{2k}/Sp(2k)_1$ coset theory or, equivalently, by the $SU(2)_k$ Wess-Zumino-Witten model. For a more detailed discussion, see Appendix B.

V. QUASIPARTICLES FROM THE PROJECTIVE CONSTRUCTION

We can understand the non-Abelian quasiparticles of the Z_k FQH states as holes in the parton integer quantum Hall states.²³ After projection, these holes become the non-Abelian quasiparticles and we can analyze these quasiparticles using either the bulk CS theory or through the edge theory/bulk wave function, all of which we obtained from the projective construction. The easiest way to analyze the quasiparticles is through the latter approach, which we describe first. The fundamental quasihole is the one with a single hole in one of the parton IQH states. We expect the wave function for this excited state to be, as a function of the quasiparticle coordinate η and the electron coordinates $\{z_i\}$,

$$\begin{aligned} \Phi_\gamma(\eta; \{z_i\}) &\sim \langle 0 | \prod_i \Psi_e(z_i) \psi_1^\dagger(z_\infty) \psi_1(\eta) | \Phi_{parton} \rangle \\ &\sim \left\langle e^{-i(N+q_1)\phi(z_\infty)} \prod_i \Psi_e(z_i) \psi_1(\eta) \right\rangle. \end{aligned} \quad (11)$$

More general quasiparticles should be related to operators of the form $\psi_i\psi_j\psi_k\cdots$. To see whether these operators really correspond to the non-Abelian quasiparticles of the Z_k parafermion states, we can study their pattern of zeros.^{13,15} The pattern of zeros is a quantitative characterization of quasiparticles in the FQH states. In general, it may not be a complete one-to-one labeling of the quasiparticles, but in the case of the Z_k parafermion states, it is; one way to see this from the projective construction approach is to compute the ground state degeneracy on the torus from the projective construction, which yields the number of topologically distinct quasiparticles, and then to observe that the number of operators with distinct pattern of zeros is the same as the number of distinct quasiparticles.

The pattern of zeros $\{l_{\gamma,a}\}$ is defined as follows.¹³ Let V_γ denote the quasiparticle operator, and let $V_{\gamma,a} = \Psi_e^a V_\gamma$. Then,

$$\Psi_e(z)V_{\gamma,a}(w) \sim (z-w)^{l_{\gamma,a+1}}V_{\gamma,a+1} + \cdots, \quad (12)$$

where \cdots represent terms higher order in powers of $(z-w)$. From $\{l_{\gamma,a}\}$ we construct the occupation number sequence $\{n_{\gamma,l}\}$ by defining $n_{\gamma,l}$ to be the number of a for which $l_{\gamma,a} = l$. The occupation number sequences $n_{\gamma,l}$ are periodic for large l and topologically distinct quasiparticles will have occupation numbers with distinct unit cells for large l . In Table I, we have listed pattern of zeros for some of the operators of the form $\psi_i\psi_j\cdots$. We see that they coincide exactly with the known quasiparticle pattern of zeros in the Z_k parafermion states, indicating that these operators do indeed correspond to the quasiparticle operators of the Z_k parafermion states. Note that two sets of operators correspond to topologically equivalent quasiparticles if either they can be related to each other by a gauge transformation or by the electron operator. In Table I, some of the gauge equivalences are indicated, using the symbol \sim . There are also various operators that are not simply gauge equivalent but that also differ by electron operators. For example, in the Z_3 states for $M=0$, the operators ψ_1 and $\psi_1\psi_2\psi_1$ are topologically equivalent quasiparticle operators; for the Z_2 states at $M=0$, ψ_i and $\psi_i\psi_j\psi_k$ are also topologically equivalent, etc.

The fundamental non-Abelian excitation in the Z_k parafermion states is the excitation that carries minimal charge and whose fusion with itself can generate all other quasiparticles. In the projective construction point of view, this operator is ψ_i , for $i=1, \dots, 2k$ (they are all gauge equivalent), and corresponds to a single hole in one of the parton IQH states. In the $M=0$ Z_k parafermion states, this operator has electromagnetic charge $Q=1/2$; its scaling dimension can be found using the stress-energy tensor of the $U(1)^{2k}/Sp(2k)_1$ theory (see Appendix B): $h_{\psi_i} = 1/2 - (2k+1)/4(k+2) = 3/4(k+2)$, which agrees with the known results. Notice that for operators with an odd number of parton fields, the $U(1)^n$ contribution to the scaling dimension is half-integer; this is related to the fermionic core that we put in by hand when we use the pure $U(M) \times Sp(2k)$ gauge theory from Eq. (4).

One way to understand how the trivial electronic excitations of the parton IQH states become non-Abelian excitations is by considering the bulk effective theory. The low energy effective theory is a theory of partons coupled to a $U(M) \times Sp(2k)$ gauge field, which implements the projection

TABLE I. We display the pattern of zeros^{13,15} $\{n_l\}$ for the various parton operators and their electromagnetic charge, Q , modulo 1. The operators ψ_i are here chiral free fermion operators in a 1+1d CFT. Normal ordering is implicit. There are many different operators that correspond to topologically equivalent quasiparticles. Here we listed the ones with minimal scaling dimension and \sim indicates gauge equivalences between various operators. The asymptotic values of the sequence $\{n_l\}$ for large l classifies each equivalence class. For the $M=0$ states, each parton operator ψ_i has electromagnetic charge $q_i=1/2$. For the $M=1$ states, ψ_i has charge $1/(k+2)$ for $i=1, \dots, 2k$ and ψ_{2k+1} has charge $k/(k+2)$.

Z_2 states, $M=0$, $\Psi_e = \psi_1\psi_2 + \psi_3\psi_4$		
Parton operators	$\{n_l\}$	$Q\%1$
Ψ_e	2 0	0
$\psi_1\psi_3 \sim \psi_1\psi_4 \sim \dots$	0 2	0
ψ_i	1 1	1/2
Z_2 states, $M=1$, $\Psi_e = \psi_5(\psi_1\psi_2 + \psi_3\psi_4)$		
Parton operators	$\{n_l\}$	$Q\%1$
Ψ_e	1 1 0 0	0
$\psi_1\psi_3 \sim \psi_2\psi_4 \sim \dots$	0 1 1 0	1/2
$\psi_1\psi_3\psi_5 \sim \psi_2\psi_3\psi_5 \sim \dots$	0 0 1 1	0
$\psi_1\psi_2 \sim \psi_3\psi_4$	1 0 0 1	1/2
$\psi_1 \sim \dots \sim \psi_4$	1 0 1 0	1/4
$\psi_1\psi_5 \sim \psi_4\psi_5 \sim \dots$	0 1 0 1	3/4
Z_3 states, $M=0$, $\Psi_e = \psi_1\psi_2 + \psi_3\psi_4 + \psi_5\psi_6$		
Parton operators	$\{n_l\}$	$Q\%1$
Ψ_e	3 0	0
ψ_i	2 1	1/2
$\psi_1\psi_3 \sim \psi_1\psi_4 \sim \psi_1\psi_5 \sim \psi_1\psi_6 \sim \dots$	1 2	0
$\psi_1\psi_3\psi_5 \sim \psi_1\psi_3\psi_6 \sim \dots$	0 3	1/2
Z_4 states, $M=0$, $\Psi_e = \psi_1\psi_2 + \psi_3\psi_4 + \psi_5\psi_6 + \psi_7\psi_8$		
Parton operators	$\{n_l\}$	$Q\%1$
Ψ_e	4 0	0
ψ_i	3 1	1/2
$\psi_1\psi_3 \sim \psi_1\psi_4 \sim \dots \sim \psi_1\psi_8 \sim \dots$	2 2	0
$\psi_1\psi_3\psi_5 \sim \psi_1\psi_3\psi_6 \sim \dots$	1 3	1/2
$\psi_1\psi_3\psi_5\psi_7 \sim \psi_1\psi_3\psi_6\psi_8 \sim \dots$	0 4	0

onto the physical Hilbert space. As we renormalize to low energies, generically a CS term will appear for the $U(M) \times Sp(2k)$ gauge field because it is allowed by symmetry. The CS term has the property that it endows charges with magnetic flux; therefore, two individual, well-separated partons carry both charge and magnetic flux in the fundamental representation of $U(M) \times Sp(2k)$. As one parton is adiabatically carried around another, there will be a non-Abelian Aharonov-Bohm phase associated with an electric charge being carried around a magnetic flux. We expect this point of view can be made more precise in order to compute directly

from the bulk theory various topological properties of the quasiparticles.

Note that in the above discussion, there are several levels of analysis: the bulk effective field theory, the edge theory, and the relation to wave functions via Eq. (11). The relation to wave functions in Eq. (11) is specific to the quasiholes; to create quasielectron excitations instead (excitations whose charge has the same sign as that of the electron), we need to create particlelike excitations using states from higher parton Landau levels and subsequently project onto the appropriate Hilbert space. Given the relation to CFT correlators in Eq. (11), it may be possible that the quasielectron operators discovered in Ref. 14 can also be written in terms of parton operators. In contrast to the wave function analysis, if the parton operators ψ_i are interpreted as operators in the edge theory corresponding to quasiholes, then ψ_i^\dagger may be viewed as operators in the edge theory that correspond to quasielectrons. Similarly, the perspective from the bulk effective field theory applies equally well to quasiholes and to quasielectrons.

VI. DISCUSSION

We conclude that the correct and most natural description of the effective field theory for the Z_k parafermion FQH states is the $U(M) \times Sp(2k)$ CS theory presented here, for which various topological properties can be explicitly computed. In this case, the role of the $U(M) \times Sp(2k)$ gauge field is clear: it is to implement the projection onto the physical Hilbert space generated by the electron operator. In particular, the $SU(2)$ quantum numbers are physical and we should now be able to couple to them through external probes in the bulk. More precisely, this means the following. The states in the physical Hilbert space, \mathcal{H}_{phys} , can all be written in terms of parton operators acting on the vacuum and a projection operator that projects onto the physical states. The states in \mathcal{H}_{phys} therefore admit an action of the group of transformations, $G = O(2M+4k)$, that act on the partons. Group elements from the $U(M) \times Sp(2k)$ subgroup of G must keep all of the physical states invariant and the low energy effective action must preserve this $U(M) \times Sp(2k)$ gauge symmetry. On the other hand, the states in \mathcal{H}_{phys} do transform under the subgroup $G \setminus U(M) \times Sp(2k)$, and this contains the $SU(2)$ subgroup that is a symmetry of the edge $U(1) \times SU(2)_k / U(1)$ theory. The low energy effective action—in terms of partons coupled to the $U(M) \times Sp(2k)$ gauge field—can now break this $SU(2)$ symmetry, as can the edge theory, e.g., by coupling to external fields.

Observe that the electron operator for the Z_k states is a sum of operators: $\Psi_e = \Psi_1 + \Psi_2 + \dots + \Psi_k$. This implies that the Z_k parafermion wave functions can actually be thought of as a (anti)symmetrization of a k layer state, $\Phi_{Z_k} = \mathcal{S}\{\Phi_{abl}(\{z_i^{(l)}\})\}$, where

$$\Phi_{abl} \sim \left\langle \prod_{i,l} \Psi_l(z_i^{(l)}) \right\rangle, \quad (13)$$

and $z_i^{(l)}$ is the coordinate of the i th electron in the l th layer. $\mathcal{S}\{\dots\}$ refers to symmetrization or antisymmetrization, depending on whether the particles are boson or fermions, re-

spectively. In the case $M=0$, Φ_{abl} is a k -layer wave function with a $\nu=1/2$ Laughlin state in each layer. For $M=1$, it is a generalized (331) wave function. The fact that the Z_k parafermion wave functions correspond to (anti)symmetrizations of these k -layer wave functions was first observed in Ref. 16.

The case $k=2$ corresponds to the Pfaffian and it is well-known that the Pfaffian wave function is equal to a symmetrization of the $(n, n, n-2)$ bilayer wave function, a fact that is closely related to the existence of a continuous phase transition between the $(n, n, n-2)$ bilayer wave function and the single-layer Pfaffian as the interlayer tunneling is increased.^{17,18} These observations suggest a myriad of possibly continuous phase transitions between various multilayer Abelian and non-Abelian states as the interlayer tunneling is tuned, which can be theoretically described by gauge-symmetry breaking. For example, breaking the $Sp(2k)$ gauge symmetry down to $SU(2) \times \dots \times SU(2)$ would correspond to a phase transition from a single-layer Z_k parafermion state to a k -layer Abelian state. Breaking $Sp(8)$ to $Sp(4) \times Sp(4)$ could correspond to a transition between the Z_4 parafermion state and a double layer state with a Pfaffian in each layer.

Finally, it is interesting to notice that the two ways of thinking about the edge theory and the quasiparticle content provide a physical manifestation of the mathematical concept of level-rank duality. On the one hand, the edge theory is a projection of free fermions by the gauge group that keeps the electron operator invariant, while on the other hand, it can be understood by considering the representation theory of the electron operator algebra. The fact that both perspectives yield the same results is a manifestation of level-rank duality.

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APPENDIX A: CALCULATION OF TORUS GROUND-STATE DEGENERACY

Here we calculate the ground state degeneracy on a torus for the $U(1) \times Sp(2k)$ Chern-Simons theory, which is the bulk effective theory for the $M=1$ Z_k parafermion states. This calculation highlights the fact that simply specifying the gauge group and the level are not enough to fully specify the bulk effective theory; one needs also to specify the allowed large gauge transformations, which can be done by specifying a choice of electron operator.

For the $M=1$ Z_k parafermion states, we take the electron operator to be

$$\Psi_e = \psi_{2k+1} \sum_{a=1}^k \psi_{2a-1} \psi_{2a}. \quad (A1)$$

The gauge field takes values in the Lie algebra of $U(1) \times Sp(2k)$, which in this case consists of $(2k+1) \times (2k+1)$ matrices: $\begin{pmatrix} T^a & 0 \\ 0 & 0 \end{pmatrix}$ and $\text{diag}(0, 1, 0, 1, \dots, 0, 1, -1)$, with T^a the generators of $Sp(2k)$ in the fundamental representation.

To compute the ground state degeneracy on a torus, we follow the procedure outlined in Ref. 4. The classical con-

figuration space of CS theory consists of flat connections, for which the magnetic field vanishes: $b = \epsilon_{ij} \partial_i a_j = 0$. This configuration space is completely characterized by holonomies of the gauge field along the noncontractible loops of the torus

$$W(\alpha) = \mathcal{P} e^{i\oint \alpha^a dl}. \quad (\text{A2})$$

More generally, for a manifold M , the gauge-inequivalent set of $W(\alpha)$ form a group: $(\text{Hom}(\pi_1(M) \rightarrow G)/G)$, which is the group of homomorphisms of the fundamental group of M to the gauge group G , modulo G . For a torus, $\pi_1(T^2)$ is Abelian, which means that $W(\alpha)$ and $W(\beta)$, where α and β are the two distinct noncontractible loops of the torus, commute with each other and we can always perform a global gauge transformation so that $W(\alpha)$ and $W(\beta)$ lie in the maximal Abelian subgroup, G_{abl} , of G (this subgroup is called the maximal torus). The maximal torus is generated by the Cartan subalgebra of the Lie algebra of G ; in the case at hand, this Cartan subalgebra is composed of $k+1$ matrices, k of which lie in the Cartan subalgebra of $Sp(2k)$, in addition to $\text{diag}(0, 1, 0, 1, \dots, 0, 1, -1)$. Since we only need to consider components of the gauge field a^I that lie in the Cartan subalgebra, the CS Lagrangian becomes

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a^I \partial a^J, \quad (\text{A3})$$

where $K_{IJ} = \text{Tr}(p^I p^J)$ and $p^I, I=1, \dots, k+1$ are the generators that lie in the Cartan subalgebra.

There are large gauge transformations $U = e^{2\pi x_i p^I/L}$, where x_1 and x_2 are the two coordinates on the torus and L is the length of each side. These act on the partons as

$$\psi \rightarrow U\psi, \quad (\text{A4})$$

where $\psi^T = (\psi_1, \dots, \psi_{2k+1})$, and they take $a^I \rightarrow a^I + 2\pi/L$. These transformations will be the minimal large gauge transformations if we normalize the generators as follows:

$$p_{ij}^I = \delta_{ij} (\delta_{i,2I} - \delta_{i,2I-1}), \quad I=1, \dots, k, \\ p^{k+1} = \text{diag}(0, 1, 0, 1, \dots, 0, 1, -1). \quad (\text{A5})$$

Thus, for example for the case $k=3$, the K matrix is

$$K = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & k+1 \end{pmatrix}. \quad (\text{A6})$$

In addition to the large gauge transformations, there are discrete gauge transformations $W \in U(1) \times Sp(2k)$ which keep the Abelian subgroup unchanged but interchange the a^I 's among themselves. These satisfy

$$W^\dagger G_{abl} W = G_{abl}, \quad (\text{A7})$$

or, alternatively,

$$W^\dagger p^I W = T_{IJ} p^J, \quad (\text{A8})$$

for some $(k+1) \times (k+1)$ matrix T . These discrete transformations correspond to the independent ways of interchanging the partons.

In this $U(1) \times Sp(2k)$ example, there are $k(k+1)/2$ different discrete gauge transformations W . k of them correspond to interchanging ψ_{2i-1} and ψ_{2i} , for $i=1, \dots, k$, and $k(k-1)/2$ correspond to the independent ways of interchanging the k different terms in the sum of Eq. (A1).

Picking the gauge $a_0^I = 0$ and parametrizing the gauge field as

$$a_1^I = \frac{2\pi}{L} X_1^I, \quad a_2^I = \frac{2\pi}{L} X_2^I, \quad (\text{A9})$$

we have

$$L = 2\pi K_{IJ} X_1^I X_2^J. \quad (\text{A10})$$

The Hamiltonian vanishes. The conjugate momentum to X_2^J is

$$p_2^J = 2\pi K_{IJ} X_1^I. \quad (\text{A11})$$

Since $X_2^J \sim X_2^J + 1$ as a result of the large gauge transformations, we can write the wave functions as

$$\psi(\vec{X}_2) = \sum_{\vec{n}} c_{\vec{n}} e^{2\pi i \vec{n} \cdot \vec{X}_2}, \quad (\text{A12})$$

where $\vec{X}_2 = (X_2^1, \dots, X_2^{k+1})$ and \vec{n} is a $(k+1)$ -dimensional vector of integers. In momentum space the wave function is

$$\phi(\vec{p}_2) = \sum_{\vec{n}} c_{\vec{n}} \delta^{k+1}(\vec{p}_2 - 2\pi \vec{n}) \sim \sum_{\vec{n}} c_{\vec{n}} \delta^{k+1}(K \vec{X}_1 - \vec{n}), \quad (\text{A13})$$

where $\delta^{k+1}(\vec{x})$ is a $(k+1)$ -dimensional delta function. Since $X_1^J \sim X_1^J + 1$, it follows that $c_{\vec{n}} = c_{\vec{n}'}$, where $(\vec{n}')^I = n^I + K_{IJ}$, for any J . Furthermore, each discrete gauge transformation W_i that keeps the Abelian subgroup G_{abl} invariant corresponds to a matrix T_i [see Eq. (A8)], which acts on the diagonal generators. These lead to the equivalences $c_{\vec{n}} = c_{T_i \vec{n}}$. The number of independent $c_{\vec{n}}$ can be computed for each k ; carrying out the result on a computer, we find that there are $(k+1)(k+2)/2$ independent wave functions, which agrees with the known torus ground state degeneracy of the Z_k parafermion states.

APPENDIX B: LEVEL-RANK DUALITY

To understand the level-rank duality better, let us examine the equivalence between the $U(1)^{2kn}$ CFT, which is the CFT of $2kn$ free fermions, and the $Sp(2k)_n \times Sp(2n)_k$ WZW model. Evidence for the equivalence of these two theories can be easily established by noting that they both have the same central charge, $c=2kn$, and that the Lie algebra $Sp(2k) \oplus Sp(2n)$ can be embedded into the symmetry group of the free fermion theory, $O(4kn)$.¹⁹ The possibility of this embedding implies that we can construct currents,

$$J^A = \frac{1}{2} \eta_\alpha T_{\alpha\beta}^A \eta_\beta, \quad J^a = \frac{1}{2} \eta_\alpha T_{\alpha\beta}^a \eta_\beta, \quad (\text{B1})$$

where the $\{\eta_\alpha\}$ are Majorana fermions, which are related to the complex fermions as $\psi_i = \eta_{2i} + i\eta_{2i+1}$. $\{T^A\}$ and $\{T^a\}$ are mutually commuting sets of $4kn \times 4kn$ skew-symmetric matrices that lie in the Lie algebra of $SO(4kn)$ and that separately generate the $Sp(2k)$ and $Sp(2n)$ Lie algebras, respectively. These currents satisfy the $Sp(2k)_n \times Sp(2n)_k$ current algebra, as can be seen by computing the OPEs

$$\begin{aligned} J^A(z)J^B(w) &\sim \frac{n\delta_{AB}}{(z-w)^2} + \frac{if_{ABC}J^C(w)}{z-w} + \dots, \\ J^a(z)J^b(w) &\sim \frac{k\delta_{ab}}{(z-w)^2} + \frac{if_{abc}J^c(w)}{z-w} + \dots, \\ J^a(z)J^A(w) &\sim O[(z-w)^0]. \end{aligned} \quad (\text{B2})$$

To compute the levels n and k , we have normalized the generators in the conventional way, so that the quadratic Casimir in the adjoint representation is twice the dual Coxeter num-

ber of the corresponding Lie algebra. The stress-energy tensor for the $Sp(2k)_n \times Sp(2n)_k$ theory, defined as

$$T(z) = \frac{1}{2(k+n+1)} \left(\sum_A J^A J^A + \sum_a J^a J^a \right), \quad (\text{B3})$$

therefore satisfies the same algebra as the stress-energy tensor of the free fermion theory: $T_{U(1)}(z) = \frac{1}{2} \sum_\alpha \eta_\alpha \partial \eta_\alpha$. Thus, for the $U(1)^{2k}/Sp(2k)_1$ edge theory of the $M=0$ Z_k parafermion states, we can take the stress tensor to be

$$T_{Z_k}(z) = T_{U(1)}(z) - \frac{1}{2(k+2)} \sum_A J^A J^A. \quad (\text{B4})$$

We can use this stress tensor to compute the scaling dimensions of the quasiparticle operators in the edge theory.

The level-rank duality should hold for general M as well, however in that case the edge theory is more complicated. It is not a g_k WZW model where g is a semisimple Lie group; instead, the edge theory can be thought of as a certain Z_2 orbifold of the Z_k parafermion CFT and a $U(1)$ Gaussian CFT.²⁴

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²¹The quasiparticle spin h is related to the quasiparticle ‘‘twist,’’ $\theta = e^{2\pi i h}$. h is equal to the scaling dimension modulo 1 of the corresponding quasiparticle operator on the edge.

²²To write the electron operator this way, we have renumbered the partons.

²³This is related to the observation in Ref. 16 that the Z_k parafermion quasihole wave functions can be obtained by symmetrizing or antisymmetrizing the quasihole wave functions of a generalized (331) state.

²⁴See, e.g., Ref. 20 for a discussion of this in the case of the Pfaffian.